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# Critical behaviour of a compressible $n$-component model with cubic anisotropy

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**Abstract.** A generalisation of the Larkin–Pikin–Sak model, in which an $n$-component order parameter $\mathbf{\Phi}$ is coupled to an elastic continuum, to the case of cubic anisotropy is studied in a renormalisation group calculation. It is shown to all orders in $\epsilon = 4 - d$ that the inclusion of cubic anisotropy in the coupling terms yields a second-order transition if and only if all $\alpha_i = 2\varphi_i - \nu d < 0$ ($i = 0, 1, 2$). Here $\varphi_0 = 1$ and $\varphi_1$, $\varphi_2$ are the cross-over scaling exponents for the variables $\phi_1\phi_2$ and $\phi_1^2 - \phi_2^2$, respectively. In this case the behaviour is that of the rigid model. If there is at least one $\alpha_i > 0$ (e.g. Ising and anisotropic XY and Heisenberg model) the absence of a stable fixed point is interpreted as a first-order transition. If the transition is close to second-order, renormalised exponents may be observed. The extension of the Fisher–Sak renormalisation of exponents $(\varphi_i/\nu)_{\text{ren}} = (\varphi_i - \alpha_i)/\nu$ is given, and an application to perovskite-type crystals is briefly discussed. The generalisation of the results to the case of lower symmetry and their relevance for other systems with non-analytic interaction is mentioned.

## 1. Introduction

The question of how the elastic degrees of freedom influence a phase transition is an old question in the theory of phase transitions. Since there is a recent very comprehensive paper on this topic (Bergman and Halperin 1976, see also de Moura *et al* 1976) which gives an excellent review about the history of this question we will not repeat their introduction and refer the reader to this paper. We only want to mention the papers which try to treat this problem by a renormalisation group (RNG) calculation and are related to the present work. The first paper in this series is that of Sak (1974) who considered an isotropic $n$-component Heisenberg model which is coupled to an isotropic elastic system by a coupling term which is likewise isotropic. Since the author used the harmonic approximation for the elastic degrees of freedom the latter can be eliminated from the partition function. This leads to an additional four-spin interaction in the Hamiltonian which is non-analytic for $k = 0$, where $k$ denotes the transferred momentum. A RNG calculation then shows, that if and only if the specific heat exponent $\alpha$ of the incompressible model is positive as in the case of the Ising model, a new 'renormalised' fixed point is the stable one. However, this fixed point is unphysical for fixed external pressure since it cannot be reached if one starts from physically allowed values of the bare coupling constants. From this fact Sak (1974) concludes a first-order transition which can be justified better if one integrates his recursion relations. In the opposite case $\alpha < 0$ (XY, Heisenberg model) the behaviour is that of an incompressible model.

Very recently, a paper by de Moura *et al* (1976) appeared, considering a model which is essentially that of Sak (1974) but which includes an anisotropy in the elastic degrees of freedom. Following the line of Sak (1974) they find for $\alpha > 0$ that the new 'renormalised' fixed point is actually stable against isotropic perturbations, but it is not stable with respect to anisotropic perturbations. From this they conclude a first-order transition. In the case of the Ising model this conclusion is supported by an independent RNG calculation of Bergman and Halperin (1976) who are able to estimate the instability temperature. For $\alpha < 0$ the behaviour is again that of the rigid model. Moreover, the authors argue that also the inclusion of non-isotropic coupling terms between the elastic and the order parameter systems would not change the results. Such coupling terms reflecting the cubic (or lower) symmetry of the crystal are particularly important in the case of systems undergoing structural transitions (Aharony and Bruce 1974, Bruce and Aharony 1975). Other RNG calculations have been performed by Wegner (1974) Rudnick *et al* (1974), Imry (1974) and Aharony (1973a). Khmel'nitzkii and Shneerson (1975) used a parquet graph approach and obtained similar results as those of de Moura *et al* (1976).

The crucial role played by the specific heat exponent $\alpha$ in all before-mentioned papers arises due to the fact that the elastic deformations couple only to the energy density of the order-parameter system. If one includes other types of couplings one must expect that the specific heat exponent will lose this dominating role.

It is the aim of this paper to show that the inclusion of such anisotropic coupling terms actually changes the picture drastically. In particular, we will consider a model with cubic anisotropy. Then it turns out that if and only if *all* $\alpha_i = 2\varphi_i - \nu d$ $(i = 0, 1, 2)$ are negative the transition is of second order. Here $\varphi_0 = 1$ and $\varphi_1$, $\varphi_2$ are the cross-over scaling exponents for the variables $\phi_1\phi_2$ and $\phi_1^2 - \phi_2^2$, respectively (Pfeuty *et al* 1974). In this case the behaviour of the system is that of the rigid model. If on the other hand at least one of the $\alpha_i$ is positive there is a new 'renormalised' fixed point reflecting the compressibility of the model. This renormalised fixed point is stable with respect to isotropic but unstable with respect to anisotropic perturbations. If the system is close to this fixed point renormalised critical behaviour will be observed. New relations between the renormalised exponents and those of the rigid model are obtained. However, since there is no stable fixed point we expect that the system undergoes a first-order transition.

After this work had been performed the present author was informed about a recent paper of Bender (1976) who considered essentially the same problem, but did not take into consideration the possible renormalisation of exponents which may be relevant for the explanation of experimental data (see § 5).

The paper is organised as follows: in § 2 we derive the effective Hamiltonian of the compressible model in the case of cubic anisotropy; § 3 is devoted to the RNG analysis *to all orders in* $\epsilon$. There the fixed points and their stability as well as the renormalisation of exponents is discussed. Since the calculations are fairly tedious, we derive some of the results in § 4 in a more heuristic manner. Thus, the reader who is mainly interested in the final results may leave out § 3. Further, § 4 includes a discussion of the critical behaviour of the compressible model.

Finally in § 5 the content of this paper is summarised and some extensions of the model and conclusions are discussed. The appendix includes an explicit $O(\epsilon^2)$ calculation based on the results of § 3.

## 2. Effective Hamiltonian

We start with a Hamiltonian consisting of three parts

$$\tilde{H} = H_{\mathrm{el}} + H_{\mathrm{m}} + H_{\mathrm{int}} \tag{2.1}$$

where $H_{\mathrm{el}}$ and $H_{\mathrm{m}}$ describe the elastic displacement field $u_\alpha(x)(\alpha = 1, \ldots, d)$ and the $\bar{n}$-component order parameter $\phi_\alpha(x)$ $(\alpha = 1, \ldots, \bar{n})$, respectively:

$$\beta H_{\mathrm{el}} = \tfrac{1}{2} \int \mathrm{d}^d x \sum_{\alpha,\beta,\gamma,\delta} c_{\alpha\beta\gamma\delta} e_{\alpha\beta}(x) e_{\gamma\delta}(x) \tag{2.2}$$

$\{c_{\alpha\beta\gamma\delta}\}$ are the components of the elastic tensor and $e_{\alpha\beta}(x)$ denotes the strain tensor

$$e_{\alpha\beta}(x) = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \sum_\gamma \frac{\partial u_\gamma}{\partial x_\alpha}\frac{\partial u_\gamma}{\partial x_\beta}\right). \tag{2.3}$$

Since we want to consider a system with cubic anisotropy there are only three different elastic moduli

$$c_{\alpha\alpha\alpha\alpha} = C_{11}, \qquad c_{\alpha\alpha\beta\beta} = C_{12}, \qquad c_{\alpha\beta\alpha\beta} = \tfrac{1}{4}C_{44}, \qquad \alpha \neq \beta. \tag{2.4}$$

Under the same symmetry $H_{\mathrm{m}}$ is given by (Wilson and Fisher 1972)

$$\beta H_{\mathrm{m}} = \int \mathrm{d}^d x \left(\frac{1}{2} r_0 \Phi^2 + \frac{1}{2}(\nabla\Phi)^2 + \tilde{u}_1^0 (\Phi^2)^2 + \tilde{u}_2^0 \sum_\alpha \phi_\alpha^4\right). \tag{2.5}$$

Here we excluded a possible cubic anisotropy in the quadratic part of the Hamiltonian for the sake of simplicity (see § 5 for a discussion).

Finally the interaction between the two fields $e_{\alpha\beta}(x)$ and $\phi_\alpha(x)$ can be put in the following form (Aharony and Bruce 1974)

$$H_{\mathrm{int}} = \int \mathrm{d}^d x \left( g_0 \frac{1}{\bar{n}} \sum_\alpha e_{\alpha\alpha} \Phi^2 + g_1 \sum_{\alpha<\beta} e_{\alpha\beta}\phi_\alpha\phi_\beta + g_2 \frac{1}{\bar{n}} \sum_\alpha e_{\alpha\alpha}(\bar{n}\phi_\alpha^2 - \Phi^2)\right). \tag{2.6}$$

Here the terms with the coefficients $g_1$ and $g_2$ reflect the cubic anisotropy of the model and represent the new feature of our approach. Because of the coupling between spin and space variables we have to choose $\bar{n} = d$.

For our further considerations it is very convenient to use a tensor representation for all coupling constants. Indeed, in the case of cubic anisotropy we can represent an arbitrary fourth rank tensor by the quantities $\lambda^j_{\alpha\beta,\gamma\delta}$ $(j = 1, 2, 3)$, where

$$\begin{aligned}
\lambda^0_{\alpha\beta,\gamma\delta} &= \frac{1}{\bar{n}} \delta_{\alpha\beta}\delta_{\gamma\delta} \\
\lambda^1_{\alpha\beta,\gamma\delta} &= \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - g_{\alpha\beta\gamma\delta} \\
\lambda^2_{\alpha\beta,\gamma\delta} &= \frac{1}{\bar{n}}(\bar{n} g_{\alpha\beta\gamma\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta})
\end{aligned} \tag{2.7}$$

and

$$g_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if } \alpha = \beta = \gamma = \delta \\ 0 & \text{otherwise.} \end{cases}$$

Note, that $\lambda^i_{\alpha\beta,\gamma\delta}$ is symmetric with respect to the permutations $\alpha\leftrightarrow\beta$, $\gamma\leftrightarrow\delta$ and $\alpha,\beta\leftrightarrow\gamma,\delta$ and obeys the relation

$$\sum_{\mu,\nu}\lambda^i_{\alpha\beta,\mu\nu}\lambda^j_{\mu\nu,\gamma\delta}=\delta_{ij}\lambda^i_{\alpha\beta,\gamma\delta}. \tag{2.8}$$

With the help of (2.7), (2.8) we get for (2.6)

$$\beta H_{\text{int}}=\int \mathrm{d}^d x\sum_{j=1}^{3} g_j\sum_{\alpha,\beta}(u(x))^j_{\alpha\beta}(\phi^2(x))^j_{\alpha\beta} \tag{2.9}$$

where

$$\begin{aligned}(u)^j_{\alpha\beta}&=\sum_{\gamma\delta}\lambda^j_{\alpha\beta,\gamma\delta}u_{\gamma\delta}\\ (\phi^2)^j_{\alpha\beta}&=\sum_{\gamma,\delta}\lambda^j_{\alpha\beta,\gamma\delta}\phi_\gamma\phi_\delta.\end{aligned} \tag{2.10}$$

Following now Larkin and Pikin (1969) we write

$$\frac{\partial u_\alpha}{\partial x_\beta}=u^0_{\alpha\beta}+\frac{1}{V}\sum_{k\neq 0}\mathrm{i}k_\beta u_\alpha(k)\,\mathrm{e}^{\mathrm{i}kx} \tag{2.11}$$

where $V$ is the volume of the system at equilibrium. Then

$$\beta H_{\text{el}}=\beta H_{\text{el}}(e^0_{\alpha\beta})+\frac{1}{2V}\sum_{k\neq 0}\sum_{\alpha,\beta}A_{\alpha\beta}(k)u_\alpha(k)u_\beta(-k)+\mathrm{O}(u^3) \tag{2.12}$$

where

$$A_{\alpha\beta}(k)=\sum_{\gamma,\delta}c_{\alpha\gamma\beta\delta}k_\gamma k_\delta. \tag{2.13}$$

Similarly, for the interaction Hamiltonian we get

$$\beta H_{\text{int}}=\beta H_{\text{int}}(e^0_{\alpha\beta})-\frac{\mathrm{i}}{V}\sum_j g_j\sum_{\alpha,\beta}(\phi^2(k))^j_{\alpha\beta}u_\alpha(-k)k_\beta \tag{2.14}$$

where

$$(\phi^2(k))^j_{\alpha\beta}=\sum_{\gamma,\delta}\lambda^j_{\alpha\beta,\gamma\beta}\int_q\phi^\gamma_q\phi^\delta_{-q+k},\qquad \int_q=\int\frac{\mathrm{d}^d q}{(2\pi)^d} \tag{2.15}$$

and $\phi^\alpha_q$ is the Fourier transform of $\phi_\alpha(x)$. $H_{\text{el}}(e^0_{\alpha\beta})$ and $H_{\text{int}}(e^0_{\alpha\beta})$ depend only on the $k=0$ component of the strain field. Treating them, we distinguish two cases.

First we leave all $e^0_{\alpha\beta}$ constant which means choosing pinned boundary conditions (case 1). Then, since $H$ is quadratic in $u_\alpha$, we may perform the integration over all $u_\alpha(k)$ in the partition function. The result can be rewritten as a new effective Hamiltonian arising from the elimination of this degrees of freedom:

$$\begin{aligned}H_{\text{eff}}&=-\frac{1}{2V}\sum_{k\neq 0}\sum_{ij}\sum_{\alpha,\beta,\gamma,\delta}g_i g_j(A^{-1})_{\alpha\gamma}k_\beta k_\delta(\phi^2(k))^i_{\alpha\beta}(\phi^2(-k))^j_{\gamma\delta}\\ &=\int_k\int_q\int_p\sum_{\substack{i,j\\ \alpha,\beta,\gamma,\delta}}\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k)\phi^\alpha_q\phi^\beta_{k-q}\phi^\gamma_p\phi^\delta_{-k-p}\end{aligned} \tag{2.16}$$

where

$$\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k)=\begin{cases}-\frac{1}{2}\sum\limits_{\mu,\nu,\lambda,\rho} g_i g_j (A^{-1}(k))_{\mu\lambda} k_\nu k_\rho \lambda^i_{\alpha\beta,\mu\nu}\lambda^j_{\gamma\delta,\lambda\rho} & \text{if } k\neq 0\\ 0 & \text{if } k=0.\end{cases} \tag{2.17}$$

Since $\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k)$ depends only on the direction of $k$, we find that the elimination of the elastic degrees of freedom leads to a non-analytic behaviour of the interaction at $k=0$.

Next, we want to consider the system under fixed external pressure (case 2). To this aim we have to add a term $-V\sum_{\alpha,\beta}\sigma_{\alpha\beta}e^0_{\alpha\beta}$ to the Hamiltonian. Here $\sigma_{\alpha\beta}$ denotes the stress tensor. Then we obtain for the $e^0_{\alpha\beta}$ dependent part of the Hamiltonian

$$\beta H_{\mathrm{el}}(e^0_{\alpha\beta})+\beta H_{\mathrm{int}}(e^0_{\alpha\beta})-V\sum_{\alpha,\beta}\sigma_{\alpha\beta}e^0_{\alpha\beta}$$
$$=\frac{1}{2}V\sum_{\substack{i\\ \alpha,\beta}}\left[c_i(e^0)^i_{\alpha\beta}(e^0)^i_{\alpha\beta}+(e^0)^i_{\alpha\beta}\left(g_i\frac{1}{V}\int(\phi^2)^i_{\alpha\beta}\,\mathrm{d}^dx-(\sigma)^i_{\alpha\beta}\right)\right] \tag{2.18}$$

where the definitions of $(e^0)^i_{\alpha\beta}$, $(\sigma)^i_{\alpha\beta}$ are the same as in (2.10). Further we made use of the representation

$$c_{\alpha\beta\gamma\delta}=\sum_i c_i\lambda^i_{\alpha\beta,\gamma\delta},\qquad \begin{aligned}c_1&=C_{11}+(d-1)C_{12}\\ c_2&=\tfrac{1}{2}C_{44}\\ c_3&=C_{11}-C_{12}\end{aligned}\qquad (\bar{n}=d) \tag{2.19}$$

for the elastic constants ($\bar{n}=d$).

Integration with respect to the $e^0_{\alpha\beta}$ now yields an additional contribution $H'_{\mathrm{eff}}$ to the effective Hamiltonian:

$$\beta H'_{\mathrm{eff}}=-\sum_{\substack{i\\ \alpha,\beta}}\frac{1}{2c_i}\left[V\left((\sigma)^i_{\alpha\beta}\right)^2-2g_i(\sigma)^i_{\alpha\beta}\int(\phi^2)^i_{\alpha\beta}\,\mathrm{d}^dx+g_i^2\frac{1}{V}\left(\int(\phi^2)^i_{\alpha\beta}\,\mathrm{d}^dx\right)^2\right]. \tag{2.20}$$

Since the first term only includes the external stress we may neglect it for our further discussions. The second term describes the interaction of the order parameter with the external stress. Putting $(\sigma)^0_{\alpha\beta}=-p\delta_{\alpha\beta}$ we obtain only a shift of the critical temperature whereas $(\sigma)^{i\neq 0}_{\alpha\beta}$ may affect the system in such a way, that only $n\leq\bar{n}$ components of the order parameter (or linear combinations of them) become critical simultaneously. Then only these $n$ components of $\mathbf{\Phi}$ have to be considered in the asymptotical region (see e.g. Nattermann and Trimper 1975).

The third term of (2.20) yields a contribution to the effective interaction $\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k=0)$ at zero momentum transfer

$$\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k=0)=-\frac{1}{2}\delta_{ij}\frac{g_i^2}{c_i}\lambda^i_{\alpha\beta,\gamma\delta}. \tag{2.21}$$

It is now convenient to subtract the angular average of $\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k\neq 0)$ from the effective interaction and put it into $\tilde{u}^0_1$ and $\tilde{u}^0_2$. To this aim we define new coupling constants $u^0_1$,

$u_2^0$, and $v^{0,ij}_{\alpha\beta,\gamma\delta}(k)$ by

$$\begin{aligned} u_1^0 &= \tilde{u}_1^0 + \frac{1}{n}\tilde{v}_0^0 + \tilde{v}_1^0 - \frac{1}{n}\tilde{v}_2^0 \\ u_2^0 &= \tilde{u}_2^0 - \tilde{v}_1^0 + \tilde{v}_2^0 \end{aligned} \tag{2.22}$$

and

$$v^{0,ij}_{\alpha\beta,\gamma\delta}(k) = \tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k) - \langle \tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k \neq 0)\rangle_{\measuredangle}. \tag{2.23}$$

Here $\langle \dots \rangle_{\measuredangle}$ denotes the angular average:

$$\langle \tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k \neq 0)\rangle_{\measuredangle} = \frac{\int \mathrm{d}\Omega_d \tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k \neq 0)}{\int \mathrm{d}\Omega_d} = \tilde{v}_i^0 \delta_{ij} \lambda^i_{\alpha\beta,\gamma\delta} \tag{2.24}$$

where $\mathrm{d}\Omega_d$ is the surface element on the $d$-dimensional unit sphere.

We are now able to write the total Hamiltonian $H = H_{\mathrm{m}} + H_{\mathrm{eff}}$ as

$$\beta H = \sum_{\alpha=1}^{n} \int_k v_2^0(k)\phi_k^\alpha \phi_{-k}^\alpha + \sum_{\alpha,\beta,\gamma,\delta}^{n} \int_k \int_q \int_p \Big(u_1^0 \delta_{\alpha\beta}\delta_{\gamma\delta} + u_2^0 g_{\alpha\beta\gamma\delta} + \sum_{i,j} v^{0,ij}_{\alpha\beta,\gamma\delta}(k)\Big)\phi_q^\alpha \phi_{k-q}^\beta \phi_p^\gamma \phi_{-p-k}^\delta \tag{2.25}$$

where

$$v_2^0(k) = \tfrac{1}{2}(r_0 + k^2) \tag{2.26}$$

i.e. we choose $e^0_{\alpha\beta}$ (or $\sigma_{\alpha\beta}$) in such a way, that only the first $n$ ($\leqslant \bar{n}$) components of $\mathbf{\Phi}$ become critical simultaneously and neglect the influence of the remaining ones. For the sake of brevity we omit this elementary discussion here and refer the reader to Aharony and Bruce (1974), Bruce and Aharony (1975).

The choice of the different conditions of fixed strain (case 1) or fixed external stress (case 2) only affect the effective interaction $v^{0,ij}_{\alpha\beta,\gamma\delta}(k)$ at zero transferred momentum

$$v^{0,ij}_{\alpha\beta,\gamma\delta}(k=0) = v_i^0 \delta_{ij}\lambda^i_{\alpha\beta,\gamma\delta} = \begin{cases} -\tilde{v}_i^0 \delta_{ij}\lambda^i_{\alpha,\beta,\gamma\delta} & \text{(case 1)} \\ -(\tilde{v}_i^0 + \frac{1}{2}g_i^2/c_i)\delta_{ij}\lambda^i_{\alpha\beta,\gamma\delta} & \text{(case 2)} \end{cases} \tag{2.27}$$

whereas $v^{0,ij}_{\alpha\beta,\gamma\delta}(k \neq 0)$ is given by (2.23) in both cases. We emphasise that it is the non-analytic behaviour of the interaction $v^{0,ij}_{\alpha\beta,\gamma\delta}(k)$ for $k \to 0$ which makes the model so interesting. The influence of this non-analytic interaction on the critical behaviour for systems with cubic or lower symmetry is the topic of the following sections.

## 3. Renormalisation group analysis

In this section we will use the skeleton graph approach to derive the RNG equations (Tsuneto and Abrahams 1973, Nattermann 1975, Ginzburg 1975). This is a self-consistent perturbation theory where the Green-function lines and the vertex parts used in a diagrammatic expansion are (partly) renormalised ones. Denoting the renormalised coupling constants by

$$\begin{aligned}&\Gamma_i(q) = 12K_d(q/\Lambda)^{\epsilon-2\eta}u_i(q/\Lambda), \qquad \Gamma_i(q=\Lambda) = u_i^0\\ &\Lambda^{ij}_{\alpha\beta,\gamma\delta}(q,k) = 12K_d(q/\Lambda)^{\epsilon-2\eta}v^{ij}_{\alpha\beta,\gamma\delta}(q,k)\\ &\Lambda^{ij}_{\alpha\beta,\gamma\delta}(q=\Lambda,k) = v^{0,ij}_{\alpha\beta,\gamma\delta}(k)\end{aligned} \tag{3.1}$$

the fixed point values $u_i^*$, $v^{ij*}_{\alpha\beta,\gamma\delta}$ describing the asymptotical behaviour $q/\Lambda \to 0$ of the coupling constants follow from

$$\left.\frac{\mathrm{d}u_i}{\mathrm{d}t}\right|_{\substack{u_i=u^*\\ \hat{v}=\hat{v}^*}} = 0, \qquad \left.\frac{\mathrm{d}v^{ij}_{\alpha\beta,\gamma\delta}}{\mathrm{d}t}\right|_{\substack{u_i=u^*\\ \hat{v}=\hat{v}^*}} = 0, \qquad t = -\ln(\Lambda/q) \tag{3.2}$$

where $\hat{v} = v^{ij}_{\alpha\beta,\gamma\delta}$. We denote $v^{ij}_{\alpha\beta,\gamma\delta}$ by a broken line separating the external legs of $\alpha$, $\beta$ and $\gamma$, $\delta$ lines. Then from the diagram structure of the skeleton graphs it is clear that a diagram with broken lines contributes to $u_i$ if and only if all broken lines are edges of closed loops (see figure 1). On the other hand diagrams to $v^{ij}_{\alpha\beta,\gamma\delta}$ include at least one broken line outside of a closed loop (see figure 2). Thus we can write

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \left.\left(\frac{\mathrm{d}u_i}{\mathrm{d}t}\right)\right|_{\hat{v}=0} + F(\langle(v_2)^p(\hat{v})^q\rangle_x) \qquad p, q = 1, 2, \ldots. \tag{3.3}$$

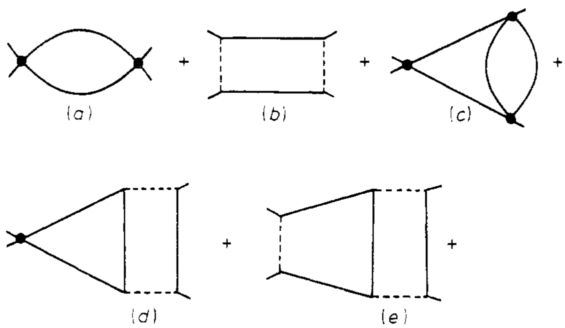


**Figure 1.** Feynman diagrams giving the lowest order contributions to the renormalised coupling constants $u_1$, $u_2$. The vertex parts $u_1$, $u_2$, and $v^{ij}_{\alpha\beta,\gamma\delta}$ are represented by full circles and by a broken line, respectively.

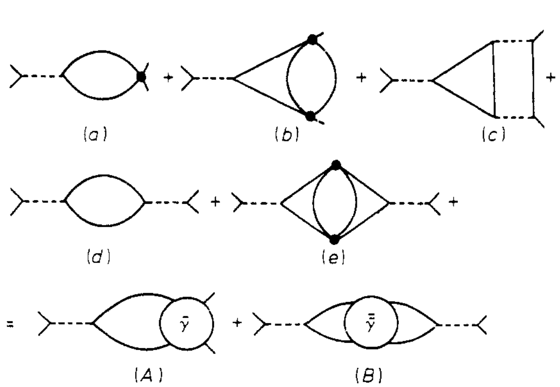


**Figure 2.** Feynman diagrams contributing to the renormalised vertex part $v^{ij}_{\alpha\beta,\gamma\delta}$. There are no terms higher than quadratic terms in $v^{ij}_{\alpha\beta,\gamma\delta}$ outside a closed loop.

Here $(\mathrm{d}u_i/\mathrm{d}t)|_{\hat{v}=0}$ and $F(\langle(v_2)^p(\hat{v})^q\rangle_{\measuredangle})$ denote the sum of all graphs without and with broken lines, respectively, contributing to $u_i$. In other words $(\mathrm{d}u_i/\mathrm{d}t)_{\hat{v}=0}=\psi(u_i)$ simply represents the RNG equation of the incompressible model. These equations are well known, therefore we will not sum them up here (see e.g. Aharony 1973b, Ketley and Wallace 1973).

The graphical representation of the equation for $v^{ij}_{\alpha\beta,\gamma\delta}(t,k)$ is depicted in figure 3. Analytically we find *to all orders in* $\epsilon=4-d$ (better: to an arbitrary number of loops)

$$\begin{aligned}\mathrm{d}v^{ij}_{\alpha\beta,\gamma\delta}(t,k)/\mathrm{d}t &= v^{ij}_{\alpha\beta,\gamma\delta}(t,k)(-\epsilon+2\eta)\\ &\quad+\sum_{\mu,\nu}(\bar{\gamma}_{\alpha\beta,\mu\nu}(t,|k|)v^{ij}_{\mu\nu,\gamma\delta}(t,k)+v^{ij}_{\alpha\beta,\mu\nu}(t,k)\bar{\gamma}_{\mu\nu,\gamma\delta}(t,|k|))\\ &\quad+\sum_{\mu,\nu,\lambda,\rho}\sum_{l}v^{il}_{\alpha\beta,\mu\nu}(t,k)\bar{\bar{\gamma}}_{\mu\nu,\lambda\rho}(t,|k|)v^{lj}_{\lambda\rho,\gamma\delta}(t,k).\end{aligned}\tag{3.4}$$

We note that the right-hand side of equation (3.4) includes only linear and quadratic terms in $v^{ij}_{\alpha\beta,\gamma\delta}$ (but of course higher order terms in $\langle(v_2)^p(\hat{v})^q\rangle_{\measuredangle}$). This is related with the fact, that the only diagrams in the Bethe–Salpeter equation for $v^{ij}_{\alpha\beta,\gamma\delta}$ which contribute to the non-analytic behaviour for $k\to0$ are ladder diagrams with respect to the broken lines. Since all vertex parts and Green functions are renormalised, the contributions of diagrams with three and more broken lines not being edges of closed loops are already included in equation (3.4). $\bar{\gamma}_{\alpha\beta,\gamma\delta}$ and $\bar{\bar{\gamma}}_{\alpha\beta,\gamma\delta}$ are given by a series of diagrams including $u_i$ and $\langle(v_2)^p(\hat{v})^q\rangle_{\measuredangle}$. Their lowest order terms follow from figure 2.

Now we write

$$\bar{\gamma}_{\alpha\beta,\gamma\delta}=\sum_{j=0}^{2}\omega_j\lambda^{j}_{\alpha\beta,\gamma\delta}\tag{3.5a}$$

and similarly:

$$\bar{\bar{\gamma}}_{\alpha\beta,\gamma\delta}=\sum_{j=0}^{2}x_j\lambda^{j}_{\alpha\beta,\gamma\delta}.\tag{3.5b}$$

Then we find from (3.4), (3.5) for the equation for $v^{ij}_{\alpha\beta,\gamma\delta}(t,k)$:

$$\begin{aligned}\mathrm{d}v^{ij}_{\alpha\beta,\gamma\delta}(t,k)/\mathrm{d}t &= v^{ij}_{\alpha\beta,\gamma\delta}(t,k)(-\epsilon+2\eta+\omega_i(t,|k|)+\omega_j(t,|k|))\\ &\quad+\sum_{\mu,\nu,l}x_l(t,|k|)v^{il}_{\alpha\beta,\mu\nu}(t,k)v^{lj}_{\mu\nu,\gamma\delta}(t,k).\end{aligned}\tag{3.6}$$

In order to get the fixed point values for the coupling constants we have to put the right-hand side of the equations (3.3) and (3.6) equal to zero.

Let us first consider the case $k\neq0$. Since the coefficients of the equations for $v^{ij}_{\alpha\beta,\gamma\delta}(k)$ are independent of $\hat{k}$ we conclude, that the fixed point solution $v^{ij}_{\alpha\beta,\gamma\delta}$ also has to be independent of $\hat{k}=k/|k|$. Hence

$$v^{ij*}_{\alpha\beta,\gamma\delta}(k\neq0)=\langle v^{ij*}_{\alpha\beta,\gamma\delta}(k\neq0)\rangle_{\measuredangle}=0\tag{3.7}$$

because of the definition of $v^{ij}_{\alpha\beta,\gamma\delta}$ (see equation (2.24)). This result implies important consequences; namely that all diagrams where broken lines are edges of closed loops vanish close to the fixed point. Therefore there are no contributions from broken lines to $u_1$, $u_2$ and thence their fixed point values are those of the rigid model.

In order to test the stability of these fixed points we have to linearise the equation for $u_i-u_i^*$. Since $v^{ij*}_{\alpha\beta,\gamma\delta}(k\neq0)=0$ the linearised right-hand side of equation (3.3) does not include terms proportional to $v^{ij}_{\alpha\beta,\gamma\delta}$. Hence also the local stability of the solutions and in particular the eigenvalue exponents are those of the rigid model. Similar from the diagram structure of the self-energy it follows that the exponents $\eta$ and $\delta$ are those of the incompressible model.

Let us now determine the fixed point values of $v^{ij}_{\alpha\beta,\gamma\delta}(k=0)=v_i\delta_{ij}\lambda^{\,i}_{\alpha\beta,\gamma\delta}$. Using (2.8) and (3.6) we get for $v_i$

$$\mathrm{d}v_i/\mathrm{d}t=v_i(-\epsilon+2\eta+2\omega_i+x_iv_i)=0 \tag{3.8}$$

where

$$\omega_i=\omega_i(t\to-\infty,k=0) \qquad x_i=x_i(t\to-\infty,k=0).$$

Hence

(i) $$v_i^*=0 \tag{3.9}$$

(ii) $$v_i^*x_i=\epsilon-2\eta-2\omega_i=\alpha_i/\nu. \tag{3.10}$$

We will call these two fixed points rigid (solution (3.9)) and renormalised (solution (3.10)).

Actually there are seven renormalised fixed points corresponding to the different combinations for $(v_0^*, v_1^*, v_2^*)$ with at least one $v_i^*\neq 0$. If we speak in the following about the rigid and the renormalised fixed point we mean by that point in particular the vanishing and non-vanishing components of $\{v_i^*\}$, respectively.

At this point we want to interrupt our RNG analysis in order to derive very useful relations between $\omega_i$ (and hence $x_iv_i^*$) and the cross-over scaling exponents $\varphi_1$, $\varphi_2$ for the variables $\phi_1\phi_2$ and $\phi_1^2-\phi_2^2$, respectively. To this aim let us add a quadratic perturbation

$$\sum_{\alpha,\beta,\gamma,\delta}\tilde{\epsilon}^{\,i}_{\gamma\delta}\lambda^{\,i}_{\alpha\beta,\gamma\delta}\int \mathrm{d}^dx\phi_\alpha\phi_\beta \tag{3.11}$$

to the Hamiltonian $H$. We begin with the remark that, at the renormalised fixed point $v_i^*\neq 0$, broken lines contribute only tree-like diagrams to the self-energy (figure 3). In this case $v^{ij}_{\alpha\beta,\gamma\delta}$ has to be taken at zero momentum.

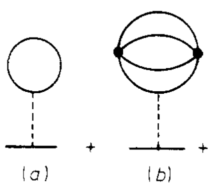

**Figure 3.** Tree-like diagrams contributing to the self-energy.

It is convenient to derive the exponents $\varphi_1$, $\varphi_2$ and the susceptibility exponent $\gamma$ from a Ward-identity for

$$\Lambda^{i}_{\alpha\beta,\lambda\rho}=\lambda^{\,i}_{\alpha\beta,\lambda\rho}-\partial\,\Sigma_{\alpha\beta}/\partial\tilde{\epsilon}^{\,i}_{\lambda\rho}|_{\tilde{\epsilon}'=0}\propto k^{(\gamma-\varphi_1)/\nu} \tag{3.12}$$

where $\Sigma_{\alpha\beta}$ denotes the self-energy part. This Ward-identity can be cast in the form (see e.g. Nattermann 1977)

$$\mathrm{d}\Lambda^{i}_{\alpha\beta,\gamma\delta}/\mathrm{d}t=\sum_{\mu,\nu}\tilde{\gamma}_{\alpha\beta,\mu\nu}\Lambda^{i}_{\mu\nu,\gamma\delta}. \tag{3.13}$$

It is now easy to see that in the *rigid case* $v_i^* = 0$ the right-hand side of (3.13) corresponds to the diagram of figure 2(*A*) if one omits there the left external legs. In other words: $\tilde{\gamma}_{\alpha\beta,\gamma\delta} = \bar{\gamma}_{\alpha\beta,\gamma\delta}$ and hence

$$\nu\omega_i = \gamma - \varphi_i. \tag{3.14}$$

The exponents $\varphi_i (i = 0, 1, 2)$ are the cross-over exponents for the perturbations $\Phi^2$, $\phi_1\phi_2$ and $\phi_1^2 - \phi_2^2$, respectively, for the rigid model. Since $\varphi_0 \equiv 1$, equation (3.14) for $i = 0$ is actually a relation between $\omega_0$ and $\gamma = \nu(2-\eta)$.

In the *renormalized case* $v_i^* \neq 0$ on the other hand, one has to include additionally the diagram of figure 2(*B*) amputating again the left external legs. Therefore $\tilde{\gamma}_{\alpha\beta,\gamma\delta} = \bar{\gamma}_{\alpha\beta,\gamma\delta} + v_i^* \bar{\bar{\gamma}}_{\alpha\beta,\gamma\delta}$ and hence

$$\omega_{i,\mathrm{r}} = \omega_i + x_i v_i^* = \epsilon - 2\eta - \omega_i. \tag{3.15}$$

This is an important relation and represents an extension of the Fisher–Sak renormalisation of critical exponents. Indeed, from (3.10), (3.14) and (3.15) we get

$$\alpha_{i,\mathrm{r}} = 2\varphi_{i,\mathrm{r}} - \nu_\mathrm{r} d = -\frac{\alpha_i \nu_\mathrm{r}}{\nu} \tag{3.16}$$

or

$$\frac{\alpha_{i,\mathrm{r}}}{\nu_\mathrm{r}} = -\frac{\alpha_i}{\nu}. \tag{3.17}$$

For $i = 0$ $\alpha_0 = \alpha$ and therefore (3.17) reduces itself to Sak's result (Sak (1974) obtained here in a diagrammatic manner. For $i \neq 0$ the result is new.

However, we note that a *renormalisation of exponents* occurs *only for those exponents* for which the corresponding *fixed point value* $v_i^* \neq 0$. For example, there is only a renormalisation of $\gamma$ (and hence of $\nu$, $\alpha$) if $v_0^* \neq 0$.

In order to decide which of the solutions (3.9), (3.10) are the stable ones we have to look for their local stability both against isotropic and anisotropic perturbations (de Moura *et al* 1976, Bergman and Halperin 1976). From (3.8) we get from a linearisation around the fixed points for $\Delta v_i = v_i - v_i^*$

$$\mathrm{d}\Delta v_i/\mathrm{d}t = \Delta v_i(-\epsilon + 2\eta + 2\omega_i + 2x_i v_i^*), \tag{3.18}$$

and hence

$$\Delta v_i = \Delta_i\, \mathrm{e}^{-\lambda_i t} = \Delta_i (\Lambda/q)^{\lambda_i} \tag{3.19}$$

where

$$\lambda_i = \begin{cases} \epsilon - 2\eta - 2\omega_i = \alpha_i/\nu & \text{if } v_i^* = 0 \quad (3.20) \\ -\epsilon + 2\eta + 2\omega_i = -\alpha_i/\nu & \text{if } v_i^* x_i = \epsilon - 2\eta - 2\omega_i. \quad (3.21) \end{cases}$$

Since we are interested in the limit $t = \ln(q/\Lambda) \to -\infty$ the solution with $\lambda_i < 0$ is stable with respect to isotropic perturbations.

In the same manner we find the *linearised* equation for anisotropic perturbations $\Delta v_{\alpha\beta,\gamma\delta}^{ij}(k \neq 0) \equiv v_{\alpha\beta,\gamma\delta}^{ij}(k \neq 0)$ (see equation (3.6))

$$\mathrm{d}\Delta v_{\alpha\beta,\gamma\delta}^{ij}(k \neq 0)/\mathrm{d}t = \Delta v_{\alpha\beta,\gamma\delta}^{ij}(k \neq 0)(-\epsilon + 2\eta + \omega_i(|k|) + \omega_j(|k|)) \tag{3.22}$$

and therefore

$$\Delta v^{ij}_{\alpha\beta,\gamma\delta}(k\neq 0)=\Delta^{ij}_{\alpha\beta,\gamma\delta}(\Lambda/q)^{\lambda_{ij}(|k|)} \tag{3.23}$$

$$\lambda_{ij}=\epsilon-2\eta-\omega_i(|k|)-\omega_j(|k|). \tag{3.24}$$

In particular, for $k<\xi^{-1}\omega_i(|k|)=\omega_i$ and therefore

$$\lambda_{ij}=\frac{1}{2}\frac{\alpha_i+\alpha_j}{\nu}. \tag{3.25}$$

From (3.20), (3.23), (3.25) it follows, that the system has a *stable fixed point* if and only if *all* $\alpha_i<0$. For $i=0$ our result agrees with that of de Moura *et al* (1976). For $i\neq 0$ the same conclusion was drawn in a recnet independent calculation of Bender (1976), however as it was already noted in § 1 this author did not take into account the possible renormalisation of the exponents as given here. An explicit $O(\epsilon^2)$ calculation of $\alpha_i/\nu$ is given in the appendix.

## 4. Critical behaviour

For the reader who is mainly interested in the final results we start in this section with an alternative derivation of some results of § 3. From a simple scaling analysis it follows, that *at the rigid fixed point* where the system behaves as an incompressible one, the renormalised vertex parts $\Gamma_i$ and $\Lambda^{ij}_{\alpha\beta,\gamma\delta}(k)$ corresponding to $u_i^0$ and $v^{0,ij}_{\alpha\beta,\gamma\delta}(k)$, respectively, scale as (see e.g. Brezin *et al* 1976)

$$\begin{aligned}&\Gamma_i(\xi)\propto\xi^{-(\epsilon-2\eta)}\\&\Lambda^{ij}_{\alpha\beta,\gamma\delta}(\xi,k)\propto\xi^{-(\omega_i+\omega_j)}.\end{aligned} \tag{4.1}$$

Here $\nu\omega_i=\gamma-\varphi_i$ and $\varphi_i (i=0,1,2)$ are the cross-over exponents for the perturbations $\mathbf{\Phi}^2$, $\phi_1\phi_2$, $\phi_1^2-\phi_2^2$, respectively, $\xi$ denotes the correlation length. Obviously, for

$$\epsilon-2\eta-\omega_i-\omega_j=\frac{\alpha_i+\alpha_j}{2\nu}<0 \tag{4.2}$$

$\Lambda^{ij}_{\alpha\beta,\gamma\delta}(\xi,k)$ vanishes faster than $\Gamma_i(\xi)$ and the rigid fixed point remains stable. On the other hand for

$$\frac{\alpha_i+\alpha_j}{\nu}>0 \tag{4.3}$$

the opposite situation appears and therefore the rigid fixed point becomes unstable. If there is a new renormalised fixed point where the vertex parts $\Lambda^{ij}_{\alpha\beta,\gamma\delta}(\xi)$ become relevant

$$(\Gamma_i(\xi))_{\rm ren}\propto(\Lambda^{ij}_{\alpha\beta,\gamma\delta}(\xi))_{\rm ren} \tag{4.4}$$

must be valid. Using the result of Sak (1974) that there is no renormalisation of $\eta$ we get $(\Lambda^{ij}_{\alpha\beta,\gamma\delta}(\xi))_{\rm ren}\propto\xi^{-\epsilon+2\eta}$. The diagrams which contribute to the renormalisation of $\Lambda^{ii}_{\alpha\beta,\gamma\delta}$ are given in figures 2(*A*) and 2(*B*). Little reflection then yields

$$(\Lambda^{ii}_{\alpha\beta,\gamma\delta}(\xi))_{\rm ren}\propto\xi^{-(\omega_i+\omega_{i,r})}, \tag{4.5}$$

where $\omega_{i,r}=\nu_r^{-1}(\gamma_r-\varphi_{ir})$ is the corresponding renormalised exponent.

From (4.4), (4.5) we obtain (our previous result equation (3.15))

$$\epsilon - 2\eta = \omega_i + \omega_{i,r} \tag{4.6}$$

Using the definitions of $\omega_i$ and $\alpha_i$ we find immediately

$$\frac{\alpha_i}{\nu} = -\frac{\alpha_{i,r}}{\nu_r} \tag{4.7}$$

and

$$\frac{\varphi_{i,r}}{\nu_r} = \frac{\varphi_i - \alpha_i}{\nu} \tag{4.8}$$

Put another way and following Sak (1974), the $k = 0$ contribution of the non-analytic interaction can be written (equations (2.25), (2.27)) as

$$v_i^0 \sum \frac{1}{V}\left(\int d^dx (\phi^2(x))^i_{\alpha\beta}{}^2\right). \tag{4.9}$$

Equation (4.9) represents a coupling between the 'energy-like' densities $[\phi^2(x)]^i$ of infinite range and can be treated by a mean field assumption. Hence we can substitute for (4.9) the expression

$$v_i^0 \epsilon_i^{\alpha\beta} \int d^dx (\phi^2(x))^i_{\alpha\beta} \tag{4.10}$$

where $\epsilon_i$ is an averaged energy-like density. This density (in the framework of the original Hamiltonian) scales as

$$\epsilon_i \propto \xi^{-d^i_{E,r}}, \qquad d^i_{E,r} = \frac{\varphi_{i,r} - \alpha_{i,r}}{\nu r}. \tag{4.11}$$

On the other hand starting with the substituted Hamiltonian (4.10) we get $(\phi^2(x))^i \alpha \xi^{-d_E}$ and thence $-d^i_E - d^i_{E,r} + d = 0$ which implies (4.6)–(4.8).

The more careful analysis of § 3 then showed, that this new renormalised fixed point is unstable against anisotropic perturbations. Thus, if at least one of the $\alpha_i = 2\varphi_i - \gamma d$ is positive, a first-order phase transition has to be expected. This result has the simple physical meaning, that there is a second-order transition only if the non-analytic behaviour of the renormalised coupling constants vanishes by approaching the critical point (i.e. if *all* $\alpha_i < 0$): $\Lambda^{ij}_{\alpha\beta,\gamma\delta}/\Sigma_i \Gamma_i \to 0$ for $\xi \to \infty$. Then the critical behaviour is that of the rigid model.

However, this condition does not seem to be realised in the most interesting cases $n = 1, 2, 3$ (i.e. Ising, XY and Heisenberg model). Indeed, considering the fixed point behaviour of $u_1$ and $u_2$ we only remind on the fact that for $n < n_c$ the isotropic Heisenberg fixed point is stable whereas for $n > n_c$ the cubic fixed point is the stable one. The value of $n_c$ depends on the order of the accuracy but at present most authors believe that for $n \leqslant 3$ the isotropic fixed point is stable (see e.g. Aharony 1973b). Then in three dimensions we have for $n = 1$ $\alpha > 0$ and for $n = 2, 3$ $\alpha < 0$ but with $\varphi_{1/2}(n = 2) = 1\cdot175$, $\nu(n = 2) = 0\cdot673$ and $\varphi_{1/2}(n = 3) = 1\cdot25$, $\nu(n = 3) = 0\cdot7$ (Pfeuty *et al* 1974) we get $\alpha_{1/2}(n = 2) = 0\cdot33$ and $\alpha_{1/2}(n = 3) = 0\cdot4$ which are positive. In other words: for the anisotropic XY and the Heisenberg model $v_0^* = 0$ but there are no stable fixed points for $v_1^*$, $v_2^*$. Previous experience with such a runaway leads to its interpretation as a first-order transition (see e.g. Nattermann and Trimper 1975, see also § 5).

## 5. Summary and conclusions

The aim of this paper has been to investigate the influence of the cubic anisotropy on the phase transition in a compressible model.

(i) In the case of cubic symmetry there are in general three different coupling terms describing the interaction between the order parameter and the elastic degrees of freedom. After eliminating the elastic deformations from the partition function additional non-analytic four-spin coupling terms appear in the Hamiltonian.

(ii) A renormalisation group calculation to all orders in $\epsilon = 4 - d$ then shows, that the transition is of second order if and only if all $\alpha_i = 2\varphi_i - \nu d < 0$ $(i = 0, 1, 2)$. Here the $\varphi_i$ denote the cross-over exponents for the perturbations $\mathbf{\Phi}^2$, $\phi_1\phi_2$, $\phi_1^2 - \phi_2^2$, respectively $(\varphi_0 \equiv 1)$. In this case the critical behaviour is that of the rigid model.

(iii) If on the other hand for some $i$ $\alpha_i > 0$ there is a renormalised fixed point $v_i^* x_i = \alpha_i/\nu$ which is stable against isotropic perturbations. Close to this fixed point $\eta$ and $\delta$ take their values of the rigid model. On the other hand, there is a renormalisation of the exponents $\varphi_i$, $\alpha_i = 2\varphi_i - \nu d$ according to

$$\frac{\varphi_{i,\mathrm{r}}}{\nu_\mathrm{r}} = \frac{\varphi_i - \alpha_i}{\nu}, \qquad \varphi_0 \equiv \varphi_{0,\mathrm{r}} \equiv 1$$

for those $i$ (and only for those) with $\alpha_i > 0$. For $i = 0$ and if $\alpha_0 \equiv \alpha > 0$ this equation determines the renormalised value $\nu_\mathrm{r}$ of $\nu$ and agrees with an earlier result of Sak (1974). However, the renormalised fixed point $v_i^* \neq 0$ is unstable with respect to anisotropic perturbations. Thus a first-order transition is expected for the Ising and anisotropic XY and Heisenberg model independently of the external conditions (i.e. constant volume or constant pressure).

(iv) Explicit results for the ratios $\alpha_i/\nu$ are given to order $\epsilon^2$.

From the discussion in § 3 it is obvious that we could use also a more complicated form for $v_2(k)$ which is compatible with the cubic anisotropy of the model, i.e. we could include anisotropic gradient terms (Nattermann and Trimper 1975, Nattermann 1976), dipolar interaction (Fisher and Aharony 1973) etc. This would not change our results concerning the stability of $v^{ij*}_{\alpha\beta,\gamma\delta}$. However, the fixed points $u_i^*$ itself may become unstable (Sokolov 1975).

Usually, the ratios $g_1/g_0$ and $g_2/g_0$ measuring the strength of the anisotropy of the interaction Hamiltonian $H_{\mathrm{int}}$ are very small for magnets, but large for crystals undergoing structural transitions (Murata 1976). Thus our results should apply in particular in the last mentioned case.

A precise assertion about the nature of the transition and the critical behaviour in the case $\alpha_i > 0$ is only possible if one integrates the recursion relations for $v_2(k)$, $u_1$, $u_2$ and $v^{ij}_{\alpha\beta,\gamma\delta}(k)$ from $q = \Lambda$ to $q = \xi^{-1}$, where $\xi$ denotes the correlation length. This is obviously a very hard task. Nevertheless it is possible to estimate roughly the region where the anisotropic perturbations become important by

$$\max_{\hat{k},i,j} \{v^{0,ij}_{\alpha\beta,\gamma\delta}(\hat{k})(\xi\Lambda)^{\lambda_{ij}}\} \approx 1. \tag{5.1}$$

If the initial values $v_i^0$ are close to the fixed point values $v_i^* \neq 0$ or if the anisotropic perturbations become important only very close to the fixed point, one expects to observe renormalised exponents before the system undergoes presumably a first-order transition.

A further remark is connected with the fact, that $v_i$ cannot change its sign under the renormalisation group iteration. This can be seen easily from the integration of equation (3.8). Thus the bare values of the coupling constants $v_i^0$ must be positive if the system should approach the renormalised fixed point $v_i^* > 0$. If $v_i^0 < 0$ but $v_i^* > 0$ this leads to a second mechanism for the occurrence of a first-order transition, a fact which was first considered by Sak (1974). For $i = 0$ one finds in case 1 $v_i^0 > 0$ and in case 2 $v_i^0 < 0$. Although we did not check this explicitly the same situation may exist for $i \neq 0$.

The renormalisation of exponents is particulary important for the exponents $\alpha_i$ where $\alpha_{i,\mathrm{r}} = -\alpha_i/(1-\alpha)$. Similarly, at a structural transition the ultrasonic attenuation diverges with the exponents $\rho_i = \alpha_i + \nu z$ (Murata 1976). Since $z_\mathrm{r} = z$ (this follows from $\eta_\mathrm{r} = \eta$) the renormalised values $\rho_{i,\mathrm{r}} = \alpha_{i,\mathrm{r}} + \nu_\mathrm{r} z$ are considerably lower than those of the rigid model. For example in $SrTiO_3$ ($n = 3$) we find with $z \approx 2$, $\alpha_0(n = 3) = -0{\cdot}1$, $\alpha_\mathrm{r} = \alpha$, $\nu_\mathrm{r} = \nu$, and hence $\rho_0 = \rho_{0,\mathrm{r}} = 1{\cdot}3$. However, since $\alpha_{1/2} = 0{\cdot}4$ and hence $\alpha_{1/2,\mathrm{r}} = -0{\cdot}4$ one gets $\rho_{1/2} = 1{\cdot}8$ but $\rho_{1/2,\mathrm{r}} = 1{\cdot}0$. If one takes for granted that one is close to the renormalised fixed point this renormalisation of exponents explains why one observes the lower value $\rho_0$ in many experiments instead of the higher one $\rho_{1/2}$ (Murata 1976). Clearly this question deserves further investigations since it is not clear under which conditions the system actually comes close to the renormalised fixed point. These conditions may change in determining the different $\rho_i$.

Another interesting application of our calculation represents the transition in improper ferroelectrics. In these substances the order parameter $\mathbf{\Phi}$ is different from the polarisation $P$. Such systems can be described by the Hamiltonian (see Khmel'nitzkii 1971)

$$H = H_\mathrm{m} + H_\mathrm{p} + H_\mathrm{int} \tag{5.2}$$

where $H_\mathrm{m}$ is given by (2.5) with $n = 2$ and

$$H_\mathrm{p} = \frac{1}{2V} \sum_{k \neq 0} \left(\chi_0^{-1} + 4\pi \frac{k_z^2}{k^2}\right) P_k P_{-k} + \chi_0^{-1} \frac{1}{V} P_0^2 \tag{5.3}$$

where $\chi_0$ denotes the bare electric susceptibility and

$$H_\mathrm{int} = c \int_k \int_q P_k \phi_q^1 \phi_{-k-q}^2. \tag{5.4}$$

The angular dependent part in (5.3) arises from the elimination of the electric field caused by the polarisation fluctuations and is characteristic for uni-axial ferroelectrics. Performing the Gaussian integration over $P_k$ in the partition function, we obtain for zero external electric field the effective Hamiltonian (2.16) with

$$\tilde{v}^{0,ij}_{\alpha\beta,\gamma\delta}(k) = \delta_{i1}\delta_{j1}\tilde{v}_1^0(k)\lambda^1_{\alpha\beta,\gamma\delta} \tag{5.5}$$

$$\tilde{v}_1^0(k) = \begin{cases} -\dfrac{1}{2}c^2 \dfrac{1}{\chi_0 + 4\pi k_z^2/k^2} & \text{if } k \neq 0 \\[2ex] -\dfrac{1}{2}c^2 \dfrac{1}{\chi_0} & \text{if } k = 0. \end{cases} \tag{5.6}$$

Thus, the elimination of the polarisation leads to a non-analytic interaction for the

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(ii) A renormalisation group calculation to all orders in $\epsilon = 4-d$ then shows, that the transition is of second order if and only if all $\alpha_i = 2\varphi_i - \nu d < 0$ $(i = 0, 1, 2)$. Here the $\varphi_i$ denote the cross-over exponents for the perturbations $\Phi^2$, $\phi_1\phi_2$, $\phi_1^2 - \phi_2^2$, respectively $(\varphi_0 \equiv 1)$. In this case the critical behaviour is that of the rigid model.

(iii) If on the other hand for some $i$ $\alpha_i > 0$ there is a renormalised fixed point $v_i^* x_i = \alpha_i/\nu$ which is stable against isotropic perturbations. Close to this fixed point $\eta$ and $\delta$ take their values of the rigid model. On the other hand, there is a renormalisation of the exponents $\varphi_i$, $\alpha_i = 2\varphi_i - \nu d$ according to

$$\frac{\varphi_{i,r}}{\nu_r} = \frac{\varphi_i - \alpha_i}{\nu}, \qquad \varphi_0 \equiv \varphi_{0,r} \equiv 1$$

for those $i$ (and only for those) with $\alpha_i > 0$. For $i = 0$ and if $\alpha_0 \equiv \alpha > 0$ this equation determines the renormalised value $\nu_r$ of $\nu$ and agrees with an earlier result of Sak (1974). However, the renormalised fixed point $v_i^* \neq 0$ is unstable with respect to anisotropic perturbations. Thus a first-order transition is expected for the Ising and anisotropic XY and Heisenberg model independently of the external conditions (i.e. constant volume or constant pressure).

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$$\max_{\hat{k},i,j} \{v_{\alpha\beta,\gamma\delta}^{0,ij}(\hat{k})(\xi\Lambda)^{\lambda_{ij}}\} \approx 1. \tag{5.1}$$

If the initial values $v_i^0$ are close to the fixed point values $v_i^* \neq 0$ or if the anisotropic perturbations become important only very close to the fixed point, one expects to observe renormalised exponents before the system undergoes presumably a first-order transition.

and at the cubic fixed point

$$\alpha_0^c/\nu^c = \frac{\epsilon}{3n}\left(4 - n - \frac{(n-1)\epsilon}{27n^2}(-19n^2 + 326n - 424)\right) \tag{A.4}$$

$$\alpha_1^c/\nu^c = \frac{\epsilon}{3n}\left(3n - 4 + \frac{\epsilon}{27n^2}(-3n^3 + 127n^2 - 530n + 424)\right) \tag{A.5}$$

$$\alpha_2^c/\nu^c = \frac{\epsilon}{3n}\left(n + 4 - \frac{\epsilon}{27n^2}(19n^3 + 131n^2 - 538n + 424)\right). \tag{A.6}$$

Using these results we remind of the restriction $n \leq d$ for $\alpha_2$ and $\alpha_3$. In three dimensions one gets from (A.2), (A.3) for $n = 2, 3$: $\{\alpha_i^H/\nu^H\} = \{-0{\cdot}08, 0{\cdot}48, 0{\cdot}48\}$ and $\{-0{\cdot}22, 0{\cdot}54, 0{\cdot}54\}$, respectively. At the cubic fixed point (equations (A.4), (A.5), (A.6)) one finds for $n = 2$ $\{\alpha_i^c/\nu^c\} = \{0{\cdot}10, 0{\cdot}10, 0{\cdot}96\}$ and for $n = 3$ $\{\alpha_i^c/\nu^c\} = \{-0{\cdot}24, 0{\cdot}51, 0{\cdot}55\}$. Note, that at both fixed points there is at least one $\alpha_i/\nu > 0$.

*Note added in proof.* In a recent paper, Murata (1977) obtained, to $O(\epsilon^2)$, the same results as in the present paper. However, he did not consider the generalisation of his results to all orders in $\epsilon$ and to lower than cubic symmetry, nor did he consider their application to other systems with non-analytic interaction.

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